M. Walker

Brownlee's Graduation Formulae Applied to United States Life Tables

BROWNLEE'S GRADUATION FORMULAE APPLIED TO UNITED STATES LIFE TABLES

BY

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THESIS

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Margaret Walker.

ENTITLED Brownlee's Graduation Formula applied to United States Life Tables.

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE OF Master of Arts.

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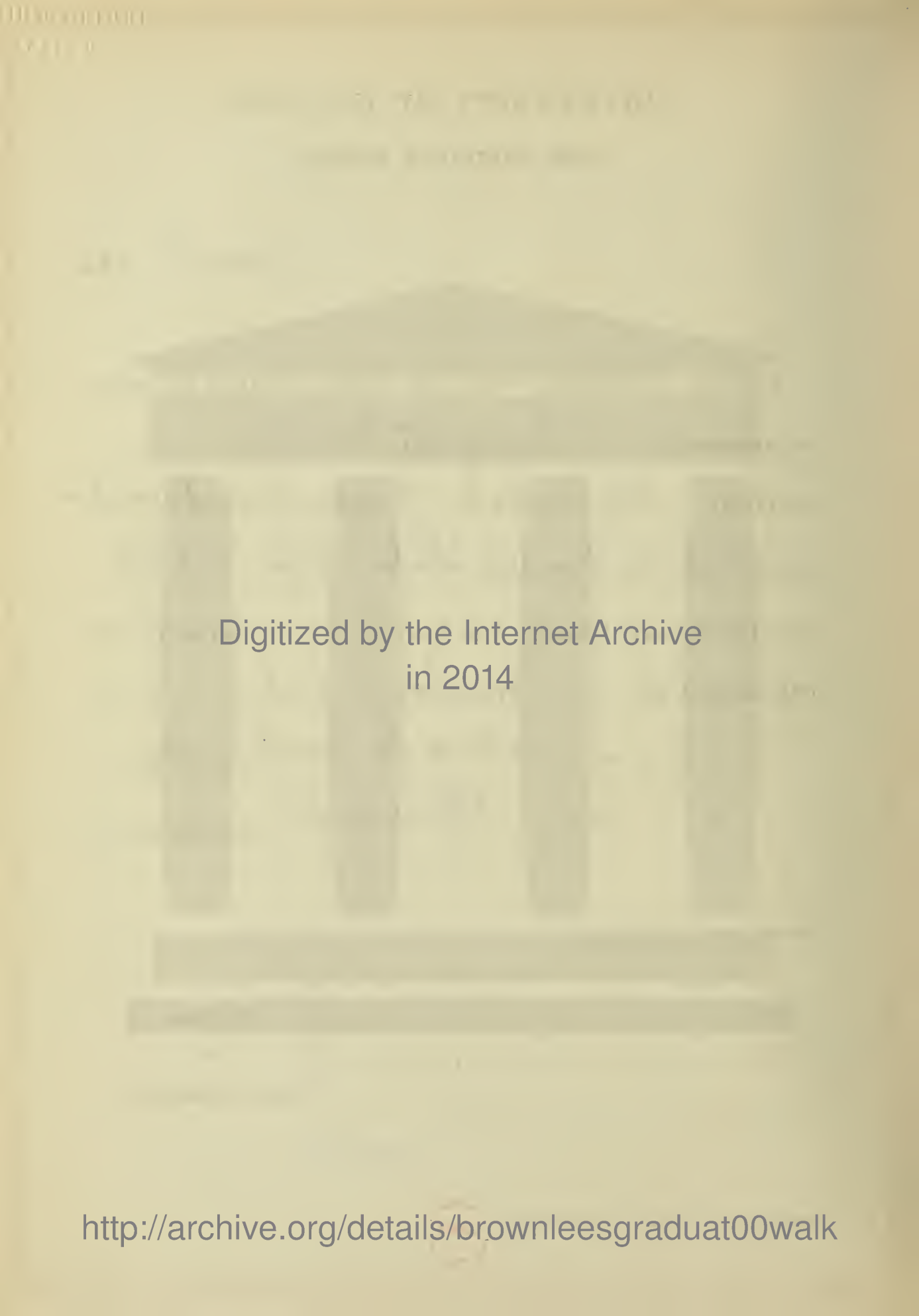
*Required for doctor's degree but not for master's



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I. INTRODUCTION

In the Journal of the Royal Statistical Society for January 1919 is an article by Doctor John Brownlee on "Notes on the Biology of a Life Table."* In this paper is discussed the results of an investigation into the graduation of certain English life tables, in which he limits himself to that portion of the tables above 10 years of age. Dr. Brownlee suggests that mortality is associated in some measure with growth, as the rate of growth decreases the mortality likewise decreases though the two have not been expressed in any mathematical relation. For example, associated with birth there is a very high death-rate which rapidly declines and reaches a minimum value at the age of 11. From this point on the rate of mortality increases to age 22 where there is a characteristic slowing up of the increase until about age 26 when it advances again more rapidly. This increase is due to physical growth and sex development. After age 40 the predominant factor in the increase of the death-rate is senescence.

Considering the expectations of life as the measure of vitality, Dr. Brownlee graduated them by a biological formula which describes the destruction of certain substances by means of ferments which combine, dissociate, and recombine and which seems analagous to the order of vital processes, growth, development, and decay. The formula used is given by the expression

$$e^{cx} = \frac{e^{a-nE}}{E}$$

where x denoted the age, E the expectation of life, e the base of

*Journal of the Royal Statistical Society, Vol. LXXXII, Part I, pages 34-78, Jan. 1919

the natural system of logarithms, and a, n, and c are constants to be determined for the various life tables.

The results of Dr. Brownlee's investigation may be summed up briefly as follows:-(1) That the values obtained by fitting the formula to the data do not actually agree with it but make a swing from one side to the other of the given values. (2) That the constants a, n, and c of the formula are rigidly related which demonstrates a relationship between all life tables. (3) That the same formula may also be used for graduating life insurance premiums.

I was interested in comparing the results obtained by applying the same formula to United States life tables with those of Dr. Brownlee and in seeing if the same conclusions could be made.

II. LIFE TABLES EMPLOYED

The data used in this paper are contained in United State Life Tables 1910 which are based upon the population taken in 1910 and the deaths occurring in the years 1909, 1910, and 1911.* These tables are the first of any scientific value prepared by the United States Government from Census returns and, being based on the general unselected population, differ from those derived from the experience of life insurance companies since the latter are based on risks selected through medical examination. The original registration states comprise Maine, New Hampshire, Vermont, Massachusetts, Rhode Island, Connecticut, New York, New Jersey, Indiana, Michigan and the District of Columbia. The estimated population in these

*Henry Moir, Sources and Characteristics of the Principal Mortality Tables, pages 26-30

states was 24, 131, 759 which is roughly one quarter of the population of the United States. The deaths occurring in the three years were as follows:

1909..... 353,576

1910..... 377,015

1911..... 368,087

Altogether twenty-five tables were published. The first is a general table dealing with both sexes and based on the total population. The second and third cover the same but there is a division into male and female lives. The next twelve are separate tables for males and females and are certain classifications of the same population which are white, negro, native white, foreign-born white, white in cities, and white in rural districts. Ten tables giving male and female mortality separately in five of the large registration states, Indiana, Massachusetts, Michigan, New Jersey, and New York, complete the report.

The following chart gives some indication of the relative mortality rates.

Table I. United States Life Tables 1910
Death Rates per 1000 of Population

Age	Native White Male	Foreign Born White Male	Negro Male	Native White Female	Foreign Born White Female
10	2.37	2.47	5.02	2.06	2.09
30	7.14	5.80	14.96	6.13	5.84
40	10.02	10.53	21.03	7.76	8.55
60	27.21	36.81	50.79	22.06	32.43
80	132.43	141.76	131.27	121.23	134.70

Corresponding Values of E

Age	Native White Male	Foreign Born White Male	Negro Male	Native White Female	Foreign Born White Female
10	51.93	50.30	40.65	54.43	52.24
30	35.61	33.71	27.33	37.98	35.31
40	28.33	26.03	21.57	30.33	27.55
60	14.58	13.06	11.67	15.78	13.59
80	5.15	4.98	5.53	5.47	5.14

Upon examination of the life tables it is found that the mortality among foreign-born white males between ages 21 to 38 is lighter than that of native white males, but at other ages is heavier. In general the mortality of men is greater than that of women. The death-rate in cities is much higher than in rural districts throughout the whole of life and negro mortality is pronouncedly higher than that of the white race. The tables dealing with the various states indicate that the death rate in the older communities, as New York and Massachusetts, is notably higher than that of Indiana and Michigan for instance. This difference is probably due to the fact that in the former are industrial centers with a large proportion of foreign-born while in the latter there is a larger proportion of rural population.

The life tables were constructed from ages 15 to 85 by the method of ocsulatory interpolation, using fifth differences. The single ages were grouped in quinquennial sets of 4 to 8, 9 to 13, 13 to 18 etc. By applying the method of osculatory interpolation the characteristic variations in the original data was less disturbed. The rates of mortality for the first five years of life were calculated by the method employed in constructing the German life tables for the ten years 1891 to 1900 and the interval from 5 to 13

was bridged over by ordinary fourth difference interpolation formulae. At the advanced ages Wittstein's formula was used, the rate of mortality being taken as unity at age 115. Spencer's 21-term formula was employed in order to join the osculatory interpolation with the Wittstein's graduation. The formula was employed over a small range sufficient to insure a smooth junction and in all cases the original data was disturbed as little as possible.

There are many factors which produce differences in the rates of mortality and they must not be overlooked. For instance, in comparing the expectations of life of people living in the country with those living in the city, the fact that cities contain a larger proportion of foreign-born and that hospitals and sanitariums are usually located in cities must be taken into account. Density of population which is associated with crime, infectious disease and ill-feeding, etc. affects the death-rate.* However, these tables are valuable as being the first of its kind prepared in the United States.

*Journal of the Royal Statistical Society, Vol.LXXXIII, Part II, pages 280-284, March 1920

In order to explain the content of the various tables, the following chart was made.

Table II. Life Tables Used in the Calculations

Table Number	Description
1	Table for both sexes in the original registration states
2	" " males in the original registration states
3	" " females in the " " "
4	Table for white males in the original registration states
5	" " " females" " " " " "
6	" " negro males " " " " " "
7	" " " females" " " " " "
8	" " native white males in the original registration states
9	" " " " females" " " " " "
10	" " foreign-born white males in the original registration states
11	" " " " females" " " " " "
12	" " white males in cities of the original registration states
13	" " " females" " " " " " "
14	" " " males in rural part of " " "
15	" " " females" " " " " " "
16	Table for males in the state of Indiana
17	" " females" " " " "
18	" " males " " " " Massachusetts
19	" " females" " " " "
20	" " males " " " " Michigan
21	" " females" " " " "
22	" " males " " " " New Jersey
23	" " females" " " " " "
24	" " males " " " " New York
25	" " females" " " " " "

III. GRADUATION OF THE EXPECTATIONS

In order to compare my results with those of Dr. Brownlee, I considered only that part of the table between 10 and 95 years of age and graduated the expectations of life, taking the values at intervals of five years. The formula, $e^{cx} = \frac{e^{a-nE}}{E}$ can be more conveniently employed by taking the logarithm upon which becomes

$$cx = a - nE - \log E \qquad (1)$$

By the method of least squares, the values of the constants a, n, and c were calculated and are given for the twenty-five tables described in Table III. Since the calculations were made by taking the logarithm to the base 10, the true values can be obtained from those in the table by multiplying by 2.3026.

Table III. Showing the Values of a, n, and c for U. S. Life Tables 1910

Table	a	n	c	Table	a	n	c
1	4.8633	.05207	.04837	16	4.0710	.03552	.03911
2	3.4001	.02653	.03159	17	4.4570	.04166	.04341
3	3.9625	.03458	.03770	18	3.5816	.02991	.03394
4	2.9476	.01803	.02635	19	4.0278	.03572	.03853
5	5.7324	.06550	.05825	20	4.1046	.03594	.03974
6	2.5433	.01756	.02116	21	4.6584	.04444	.04575
7	2.6306	.01787	.02145	22	3.6114	.03116	.03427
8	4.0078	.03662	.03851	23	3.4981	.02680	.03245
9	3.6036	.02731	.03340	24	3.4450	.02898	.03224
10	3.5235	.02971	.03337	25	3.7413	.03158	.03541
11	2.7333	.01366	.02407				
12	3.1709	.02397	.02934				
13	2.9514	.01762	.02635				
14	5.1587	.05383	.05154				
15	5.1845	.05270	.05182				

On plotting the values of these constants on squared paper it is found that for all life tables investigated the constants c and a lie on a straight line which is shown in Diagram I. The relation-

ships between a and n, and c and n are not so simple.

After obtaining the values of the constants for the various life tables, the theoretical expectations were computed by tran forming equation (1) to the form

$$u + \log u = \log n + a - cx \qquad (2)$$

where u equals nE. The results of fitting the formula to the expectations are shown in Table IV. It will be noticed from the synopsis of the twenty-five tables that there is little difference at any point between the actual and theoretical results. For instance, consider the expectations for age 25 which range from 30 years upwards. With the exception of Table VII, which is a table of expectations of life for negro females and probably inaccurate, in no case does the difference between the theoretical and actual values exceed .28 of a year which is less than one percent. In the majority of cases the difference is very much smaller, fourteen out of the twenty-five values affecting the second decimal place alone and one value, that in Table II, exactly equalling the actual expectation. The same correspondence is seen at all ages until the highest are reached where the differences increase in proportion to the actual values and amount to more than ten percent of the other values, although the actual difference in no case is greater than .5 of a year. Diagrams II and III show the comparison of the expectations of the life table with those obtained by the formula between ages 40 and 75 for life tables 9 and 24 respectively. These two tables are chosen as representative of the worst and of the best results respectively obtained by fitting the formula to the data. The red curve illustrates the formula values and the black curve the life table values.

Diagram I. Showing Relation Between a and c for U. S. Life Tables 1910

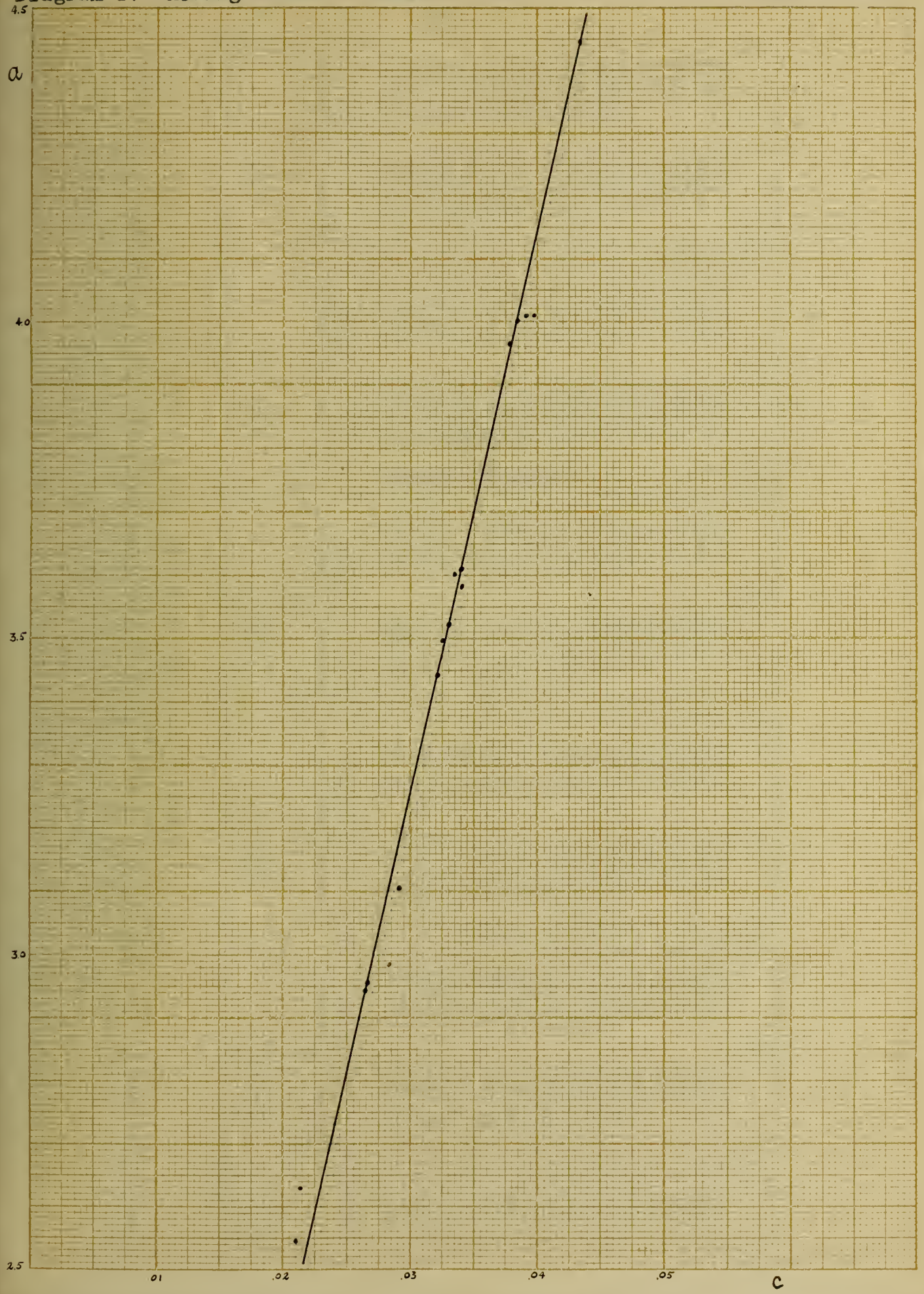


Table IV. Showing the Life Table and the Theoretical Expectations for United States Life Tables 1910. The positive sign indicates that the Life Table Value is higher than the Theoretical, and the negative sign the opposite.

Age	Table 1			Table 2			Table 3			Table 4			Table 5		
		Actual	Theoretical		Actual	Theoretical		Actual	Theoretical		Actual	Theoretical		Actual	Theoretical
10	+	52.15	51.27	-	51.07	51.67	-	53.31	53.67	-	51.32	53.16	+	53.57	52.37
15	+	47.73	47.30	-	46.66	47.20	-	48.87	49.29	-	46.91	48.21	+	49.12	48.42
20	+	43.53	43.38	-	42.48	42.84	-	44.66	44.98	-	42.71	43.42	+	44.88	44.55
25	+	39.60	39.51		38.59	38.59	-	40.69	40.76	-	38.79	38.82	+	40.88	40.71
30	-	35.70	35.71	+	34.70	34.48	+	36.79	36.65	+	34.87	34.43	+	36.96	36.91
35	-	31.90	31.98	+	30.94	30.52	+	32.95	32.65	+	31.08	30.28	-	33.09	33.19
40	-	28.20	28.35	+	27.32	26.74	+	29.15	28.78	+	27.43	26.32	-	29.26	29.50
45	-	24.54	24.81	+	23.77	23.15	+	25.36	25.07	+	23.86	22.61	-	25.45	25.91
50	-	20.98	21.40	+	20.32	19.77	+	21.67	21.53	+	20.39	19.46	-	21.74	22.42
55	-	17.55	18.14	+	16.98	16.64	-	18.13	18.19	+	17.03	16.13	-	18.18	18.91
60	-	14.42	15.05	+	13.95	13.78	-	14.90	15.09	+	13.98	13.36	-	14.92	16.49
65	-	11.60	12.17	+	11.24	11.21	-	11.96	12.25	+	11.25	10.92	-	11.97	12.81
70	-	9.11	9.54	-	8.83	8.94	-	9.38	9.71	+	8.83	8.79	-	9.38	9.99
75	-	6.99	7.23	-	6.75	7.00	-	7.20	7.51	-	6.75	7.00	-	7.20	7.47
80		5.25	5.25	-	5.10	5.37	-	5.37	5.66	-	5.09	5.50	+	5.35	5.30
85	+	4.00	3.65	-	3.90	4.05	-	4.08	4.12	-	3.88	4.27	+	4.06	3.53
90	+	3.03	2.42	+	3.01	3.00	+	3.05	2.94	-	2.99	3.23	+	3.00	2.28

Table IV. Showing the Life Table and the Theoretical Expectations for United States Life Tables 1910 - continued

Age	Table 6			Table 7			Table 8			Table 9			Table 10		
		Actual	Theoretical		Actual	Theoretical		Actual	Theoretical		Actual	Theoretical		Actual	Theoretical
10	-	40.65	40.96	-	42.84	43.51	-	51.93	52.05	-	54.43	55.77	+	50.30	50.14
15	-	36.77	37.51	-	39.18	39.72	-	47.52	47.81	-	49.98	50.33	+	45.89	45.84
20	-	33.46	33.55	+	36.14	35.47	-	43.32	45.14	-	45.76	46.45	+	41.75	41.62
25	+	30.44	30.17	+	32.97	32.55	-	39.43	39.54	-	41.84	41.95	+	37.77	37.58
30	+	27.33	26.96	+	29.61	29.20	+	35.61	35.54	+	37.98	37.59	+	33.71	33.51
35	+	24.42	23.91	+	26.44	26.01	+	31.93	31.58	+	34.15	33.37	+	29.76	29.71
40	+	21.57	21.48	+	23.34	23.00	+	28.33	27.90	+	30.33	29.31		26.03	26.03
45	+	18.85	18.38	+	20.43	20.18	+	24.74	24.29	+	26.51	25.44	-	22.46	22.52
50	+	16.21	15.92	+	17.65	17.56	+	21.20	20.85	+	22.78	21.79	-	19.08	19.22
55	+	13.82	13.67	-	14.98	15.14	+	17.75	17.59	+	19.16	18.38	-	15.87	16.15
60	+	11.67	11.64	-	12.78	12.95	+	14.58	14.57	+	15.78	15.24	-	13.06	13.34
65	-	9.74	9.82	-	10.82	10.97	-	11.67	11.91	+	12.62	12.41	-	10.58	10.80
70	-	8.00	8.88	+	9.22	9.21	-	9.09	9.34	+	9.80	9.53	-	8.40	8.57
75	-	6.58	7.44	-	7.55	7.67	-	6.86	7.18	-	7.43	7.72	-	6.54	6.65
80	-	5.53	6.19	-	6.05	6.33	-	5.15	5.37	-	5.47	5.89	-	4.98	5.05
85	-	4.48	5.07	-	5.09	5.18	-	3.89	3.90	-	4.12	4.48	+	3.86	3.76
90	-	4.01	4.18	+	4.50	4.21	+	2.97	2.76	-	3.02	3.23	+	3.02	2.74

Table IV. Showing the Life Table and the Theoretical Expectations for United States Life Tables 1910 - continued

Age	Table 11			Table 12			Table 13			Table 14			Table 15		
		Actual	Theoretical		Actual	Theoretical		Actual	Theoretical		Actual	Theoretical		Actual	Theoretical
10	-	52.24	55.04	-	49.13	49.39	-	52.22	54.16	+	54.53	54.07	+	55.54	55.45
15	-	47.79	49.57	-	44.72	44.97	-	47.77	49.09	+	50.10	49.92	-	51.07	51.19
20	-	43.50	44.31	-	40.51	40.67	-	43.51	44.21	+	45.92	45.83	+	46.86	46.98
25	+	39.34	39.31	+	36.54	36.51	-	39.46	39.50	+	42.06	41.78	+	42.95	42.83
30	+	35.31	34.58	+	32.61	32.49	+	35.52	35.00	+	38.10	37.80	+	39.05	38.74
35	+	31.39	30.14	+	28.87	28.65	+	31.67	30.73	+	34.14	33.90	+	35.10	34.73
40	+	27.55	26.01	+	25.32	25.00	+	27.88	26.76	+	30.20	30.07	+	31.15	30.80
45	+	23.75	22.22	+	21.89	21.56	+	24.14	22.96	-	26.27	26.35	+	27.18	26.98
50	+	20.09	18.77	+	18.59	18.36	+	20.53	19.51	-	22.43	22.75	-	23.27	23.28
55	+	16.63	15.68	+	15.45	15.41	+	17.10	14.19	-	18.68	19.29	-	19.47	19.72
60	+	13.59	12.95	-	12.68	12.74	+	14.04	13.54	-	15.23	16.01	-	15.93	16.35
65	+	10.93	10.58	-	10.26	10.36	+	11.32	11.06	-	12.10	12.94	-	12.64	13.20
70	+	8.67	8.54	-	8.14	8.28	+	8.99	8.91	-	9.36	10.13	-	9.76	10.32
75	-	6.78	6.83	-	6.33	6.52	-	6.99	7.08	-	7.02	7.63	-	7.38	7.75
80	-	5.14	5.41	-	4.91	5.04	-	5.31	5.56	-	5.20	5.49	-	5.40	5.57
85	-	3.95	4.30	-	3.83	3.84	-	4.08	4.32	+	3.91	3.76	+	4.05	3.80
90	-	3.07	3.32	+	3.03	2.89	-	3.11	3.32	+	2.96	2.45	+	2.94	2.46

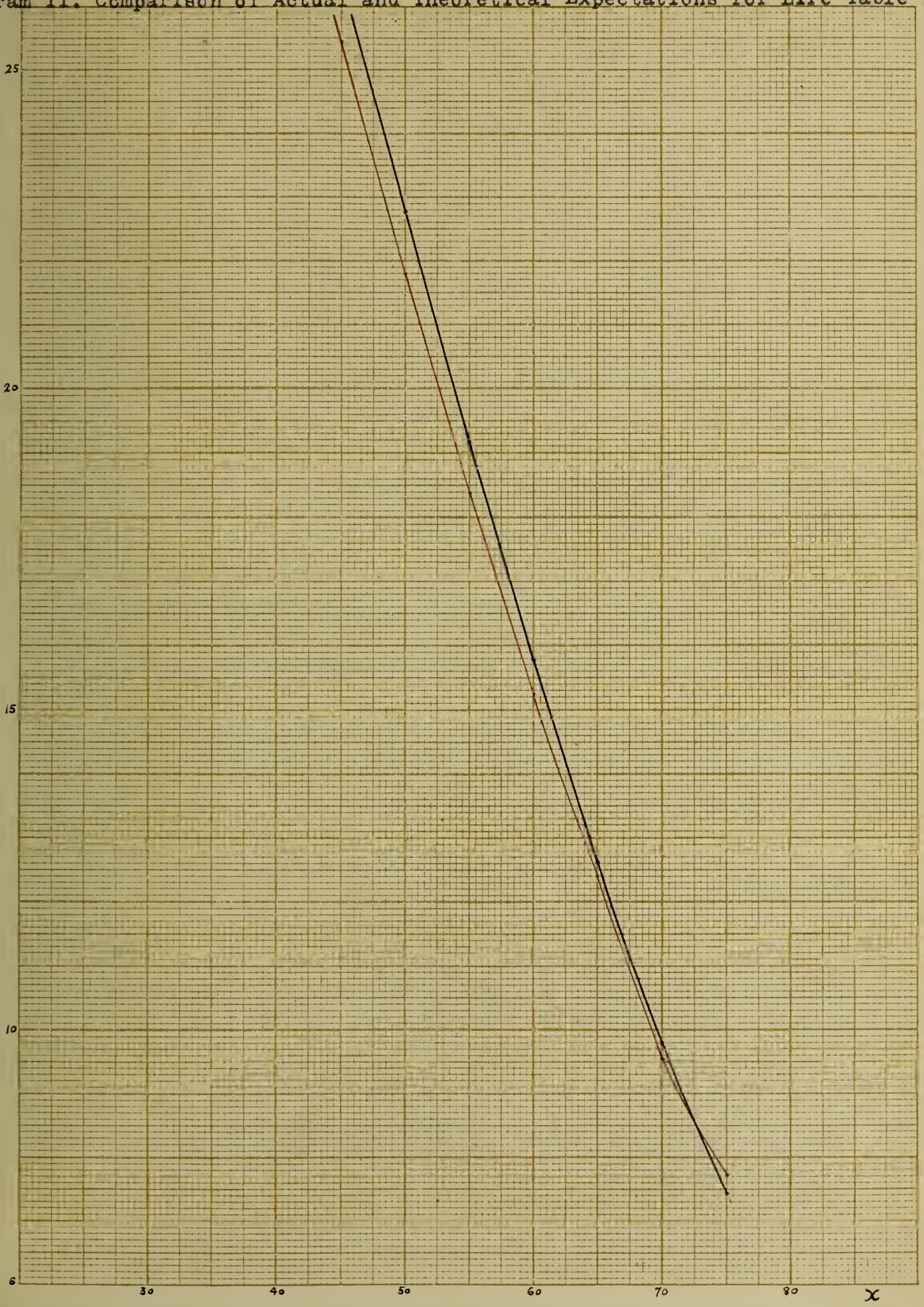
Table IV. Showing the Life Table and the Theoretical Expectations for United States Life Tables 1910 - continued

Age	Table 16		Table 17		Table 18		Table 19		Table 20	
	Actual	Theoretical	Actual	Theoretical	Actual	Theoretical	Actual	Theoretical	Actual	Theoretical
10	- 53.91	54.68	- 54.18	54.82	- 51.14	51.24	- 53.56	53.57	- 54.09	54.78
15	- 49.53	50.21	- 49.77	50.47	- 46.71	46.86	- 49.11	49.21	- 49.72	50.28
20	- 45.44	45.82	- 45.73	46.19	- 42.48	42.58	- 44.85	44.92	- 45.57	45.86
25	+ 41.66	41.52	+ 42.01	41.98	+ 38.51	38.41	+ 40.77	40.73	+ 41.71	41.53
30	+ 37.76	37.11	+ 38.29	37.85	+ 34.55	34.35	+ 36.78	36.62	+ 37.76	37.30
35	+ 33.89	33.24	+ 34.51	33.81	+ 30.72	30.44	+ 32.90	32.63	+ 33.78	33.19
40	+ 29.99	29.28	+ 30.66	29.89	+ 26.97	26.68	+ 29.04	28.77	+ 29.81	29.20
45	+ 26.14	25.48	+ 26.78	26.09	+ 23.34	23.10	+ 25.25	25.06	+ 25.85	25.66
50	+ 22.58	21.85	+ 22.93	22.45	+ 19.79	19.72	+ 21.55	21.52	+ 22.10	21.69
55	+ 18.63	18.43	+ 19.22	18.99	- 16.45	16.57	- 17.99	18.17	+ 18.42	18.28
60	+ 15.25	15.24	- 15.68	15.74	- 13.42	13.68	- 14.79	15.06	- 14.99	15.02
65	- 12.09	12.33	- 12.43	12.73	- 10.81	11.07	- 11.94	12.22	- 11.90	12.15
70	- 9.29	9.72	- 9.56	10.02	- 8.58	8.78	- 9.49	9.67	- 9.17	9.54
75	- 6.94	7.46	- 7.24	7.64	- 6.65	6.80	- 7.30	7.45	- 6.89	7.28
80	- 5.24	5.56	- 5.25	5.62	- 5.07	5.15	- 5.49	5.58	- 5.09	5.39
85	+ 4.05	4.02	- 3.97	3.99	+ 3.88	3.82	+ 4.17	4.05	- 3.85	3.87
90	+ 3.06	2.82	+ 3.00	2.73	+ 2.96	2.78	+ 3.05	2.87	+ 2.97	2.70

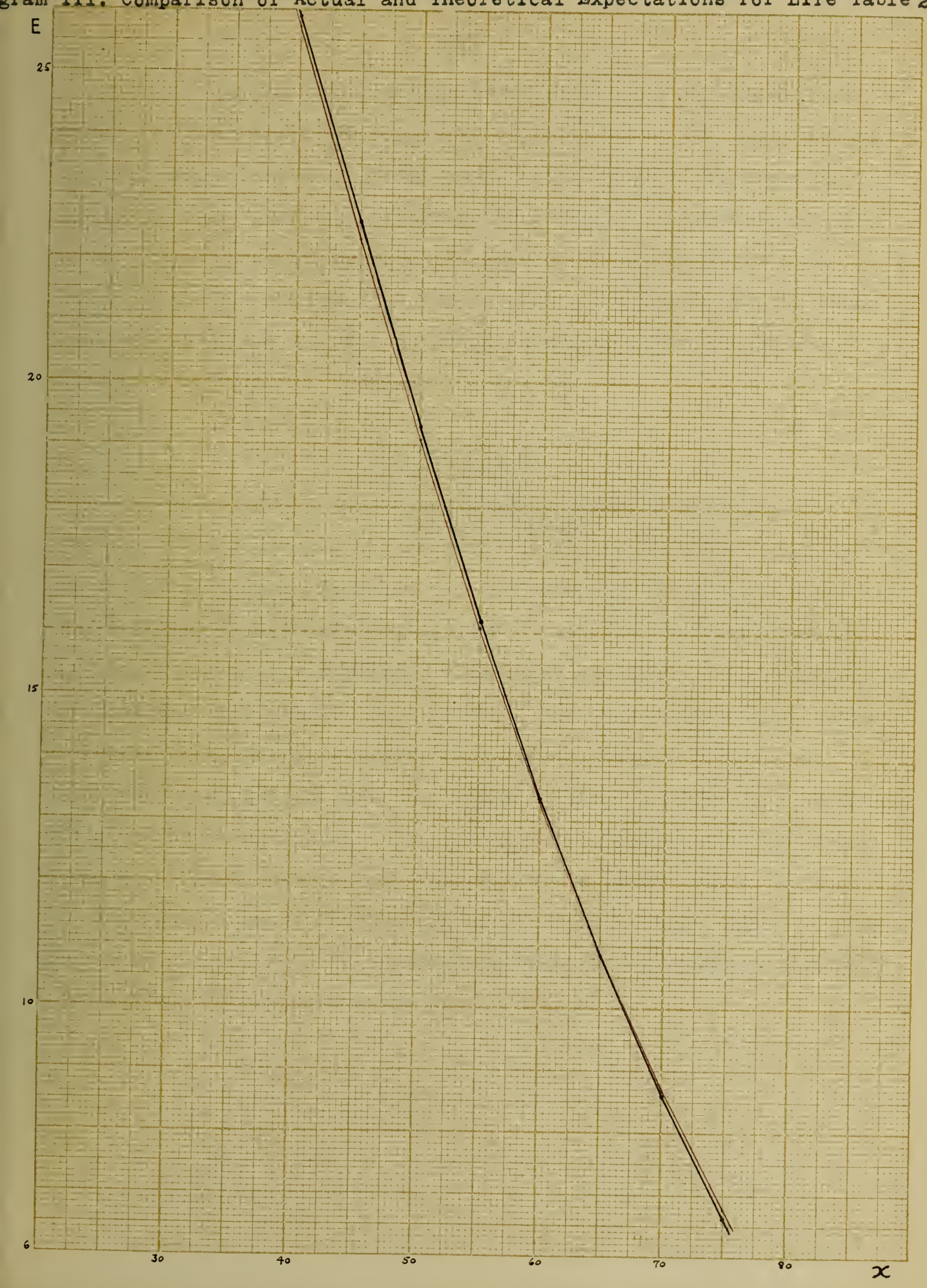
Table IV. Showing the Life Table and the Theoretical Expectations for United States Life Tables 1910 - continued

Age	Table 21			Table 22			Table 23			Table 24			Table 25		
		Actual	Theoret-ical		Actual	Theoret-ical		Actual	Theoret-ical		Actual	Theoret-ical		Actual	Theoret-ical
10	-	54.97	55.31	+	50.31	50.29	-	53.03	53.83	+	49.40	49.33	-	52.51	52.73
15	-	50.53	50.97	-	45.89	46.03	-	48.63	49.22	-	44.98	45.11	-	48.05	48.32
20	-	46.36	46.67	-	41.66	41.86	-	44.38	44.72	-	40.79	40.98	-	43.79	44.00
25	+	42.54	42.45	-	37.77	37.78	+	40.37	40.35	-	36.87	36.96	-	39.75	39.78
30	+	38.65	38.31	+	33.86	33.83	+	36.44	36.09	-	33.01	33.07	+	35.83	35.68
35	+	34.75	34.26	+	30.12	30.00	+	32.54	31.99	+	29.34	29.31	+	32.01	31.67
40	+	30.87	30.31	+	26.57	26.32	+	28.76	28.06	+	25.88	25.71	+	28.25	27.86
45	+	26.90	26.48	+	23.07	22.82	+	24.93	24.32	+	22.53	22.29	+	24.54	24.20
50	+	23.05	22.80	+	19.67	19.50	+	21.21	20.80	+	19.28	19.07	+	20.94	20.72
55	+	19.34	19.28	-	16.37	16.41	+	17.72	17.53	+	16.17	16.07	+	17.49	17.47
60	-	15.83	15.98	-	13.50	13.57	+	14.57	14.52	+	13.35	13.32	-	14.36	14.46
65	-	12.57	12.78	-	10.91	10.99	-	11.68	11.81		10.84	10.84	-	11.56	11.73
70	-	9.67	10.13	-	8.65	8.72	-	9.26	9.42	-	8.58	8.66	-	9.11	9.31
75	-	7.34	7.68	-	6.59	6.77	-	7.09	7.36	-	6.62	6.77	-	7.05	7.21
80	-	5.47	5.61	-	5.05	5.12	-	5.34	5.64	-	5.01	5.19	-	5.26	5.45
85	+	3.98	3.93	+	3.84	3.80	-	4.10	4.23	-	3.82	3.90	+	4.03	4.02
90	+	2.98	2.65	+	2.85	2.76	+	3.29	3.12	+	3.05	2.88	+	3.05	2.90

agram II. Comparison of Actual and Theoretical Expectations for Life Table 9



gram III. Comparison of Actual and Theoretical Expectations for Life Table 24



In examining the differences between the actual and theoretical values, which are indicated in Table IV by the positive and negative signs according as the value given by the life table is greater or less than that given by the formula, in the majority of cases it is found that between the ages of 10 and 25 there is a defect of the actual from the theoretical. Between 30 and 55 there is generally an excess while between 65 and 80 there is again a defect.

IV. CONCLUSIONS

In summing up the conclusions reached by applying the formula, which describes the rate of decay of substances subject to the action of organic ferments, to the expectations of life given in the United States Life Tables 1910, I may say that the results obtained compare very favorably with those obtained by Dr. Brownlee for English Life Tables. On the whole, the graduation may be taken as satisfactory for most practical purposes, though it does not completely describe the data but makes a swing from one side to the other of the given values. The same relation between the constants a and c is found, namely that they be on a straight line. It will be extremenly interesting to find if the formula would apply to the mortality curves of other countries as well as it seemed to fit English and United States mortality.